Indian Statistical Institute
End Semestral Examination- B.Math-I
Algebra-II (Backpaper)

Time : 3 hours   Max. Marks : 100

Answer question **1** and any **eight** from the rest. Question 1 carries 20 marks. Rest all carry equal marks.

(1) Prove the following statements.
(a) Let V be an n-dimensional vector space and let T be a linear operator on V. Suppose that there exists some positive integer k so that $T^k = 0$. Then $T^n = 0$.
(b) Similar matrices have the same rank.
(c) Eigenvalues of a hermitian matrix are real.
(d) Every eigenvalue of a unitary operator on a finite dimensional complex vector space has absolute value 1.

(2) Suppose that $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the linear transformation whose matrix relative to the standard ordered basis is $A = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -2 & 0 \\ -4 & 1 & 2 \end{bmatrix}$. Compute the matrix of T with respect to the ordered basis $\mathcal{B} = \{(1,-1,1),(1,-2,2),(1,-2,1)\}$.

(3) Find the eigenvalues and a basis for the eigenspaces of the matrix $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$.

(4) Let T be a linear operator on a finite-dimensional vector space V over $\mathbb{C}$. Prove that T is diagonalizable if and only if T is annihilated by some polynomial over $\mathbb{C}$ which has distinct roots.

(5) Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$, each vector of which is an eigenvector of T.

(6) Let $\langle , \rangle$ be a positive definite symmetric form on a finite dimensional real vector space V. Prove that there exists an orthonormal basis for V.

(7) Prove that a real symmetric $n \times n$ matrix A is positive definite if and only if the determinant det A_i is positive for each $i = 1, \ldots, n$, where A_i denotes the upper left $i \times i$ submatrix of A.

(8) Let V denote the vector space of real $n \times n$ matrices. Prove that $\langle A, B\rangle =$ trace (A^tB) is a positive definite bilinear form on V. Find an orthonormal basis for this form.

(9) State and prove Sylvester's law for a symmetric bilinear form on a real vector space.

(10) Prove that A is positive definite hermitian if and only if $A = P^*P$ for some invertible matrix P. Hence show that the only matrix which is both positive definite hermitian and unitary is the identity I.

(11) Prove that a complex matrix M is normal if and only if there is a unitary matrix P such that PMP^* is diagonal.